The Full Survey on The Euclidean Steiner Tree Problem

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Abstract

The Steiner Tree Problem is a famous and long-studied problem in combinatorial optimization. However, the best heuristics algorithm so far is inferior efficient only to solve some limited cases in reasonable time. This paper introduces actual solutions in some special cases and an improved heuristics algorithm based on polygon reduction method. The experimental results show the obvious superior performance over the other approaches.

Keywords: Steiner tree problem, full steiner topology, Steiner polygon reduction

1 Introduction

Consider a undirected connected graph $G = (V, E)$, where V is the set of vertices (terminal nodes), E is the set of edges in G. Let $S \subseteq V$ be a subset of distinguished vertices of V, which we shall call Steiner points. The Steiner Tree Problem (STP) is to find $|S| = \sum_{s \in S} |s|$ is a minimum. The STP has shown to be NP-Hard[2] and has attracted a considerable attention in the literature. An obvious application of STP is to finding telecommunication links, the shortest routing of services in building, and the design of road system. A variant of Steiner problem is the rectilinear Steiner problem (in the Manhattan metric) may be applied to circuit layout on computer chips. Some well-known basic properties of SMTs are as follows.

- Angles between the arcs connecting the nodes to its adjacent nodes are all 120°.
- Each Steiner point of an Steiner tree is of degree exactly three.
- An Steiner tree for n nodes contains at most $n-2$ Steiner points.

The Melzak algorithm[3] was the first algorithm for giving a finite number of steps of solution, based on the enumeration of all Steiner topologies. However, since the number of subsets and topologies pairs is super-exponential, this basic framework has limited practical value. *Weng*[4] proposed a more efficient algorithm for the construction of a

Steiner tree in the hexagonal coordinate system. In the mean time, the idea of Steiner approximation algorithm was introduced by *Karpinski* and *Zelikovsky*.[5] They obtained a performance ratio of 1.644 with an algorithm which minimizes the weighted sum of the length and the loss of a steiner tree. The best approximation ratio so far is $1+\frac{\log 3}{2} \approx 1.55$ which was presented by *Robins* and *Zelikovsky*[6] that combined the loss of a Steiner tree into the relative greedy algorithm.

2 Steiner Topology

2.1 The Number of Steiner Topologies

Full Steiner topology is an important subset of Steiner topologies. A full Steiner topology for n nodes has exactly $n-2$ Steiner points. The full Steiner topology with $n+1$ nodes can be founded as adding one more Steiner point in one of which $2n-3$ edges. An expression for the number of full Steiner topologies is,

$$f(n+1) = (2n-3)f(n)$$

which has the solution,

$$f(n) = \frac{(2n-4)!}{2^{n-2}(n-2)!}.$$

Let $F(n,k)$ denote the number of different topologies with n nodes and $k \leq n-2$ Steiner points with no nodes of degree 3, and given by,

$$F(n,k) = \binom{n}{k+2} f(k) \frac{(n+k-2)!}{(2k)!}.$$

We now consider the degree of nodes of three, n_3. The summations are obtained from Steiner topologies with $n-n_3$ nodes and $k+n_3$ Steiner points. Then, we have,

$$F(n) = \sum_{k=0}^{n-2} \sum_{n_3=0}^{\lfloor (n-k-2)/2 \rfloor} \binom{n}{n_3} F(n-n_3, k+n_3) \frac{(k+n_3)!}{k!}.$$

2.2 Full Steiner Topology Partition

For every original topology, it can be decomposed into separate full Steiner topologies. Let T be the SMT for the set of n points $\mathcal{N}$. For some divisions (n_i) of $\mathcal{N}$, where $\cup_{i=1}^{t} n_i = \mathcal{N}, |n_i| \geq 2 \, \forall \, i \in [1, n-1]$. Then, the set n_i is the partition of the division.

The topologies partition provides an alternative method for constructing a SMT, constructing all FSTs for all FTPs of every subset of $\mathcal{N}$ rather than finding all Steiner trees for all Steiner topologies. Although finding all FSTs is less work considering finding all Steiner topologies, finding all FSTs is still an extremely difficult task.

3 Steiner Polygon

From the last chapter, we can obviously find that $f(n)$ grows extremely fast. The rapid growth is the bane of optimal algorithms for the Euclidean Steiner tree problem. Therefore, it needs a efficient criterion for decomposing a Euclidean Steiner tree into a smaller problem.

3.1 Steiner Polygon Criterion

A given Steiner polygon allows an Steiner tree algorithm to constrict and relieve its computation within a given area. Therefore, we want the Steiner polygon as small as possible. The following two properties[1] may give some useful information to optimize the construction of Steiner polygon, as illustrated in Figure 1 and 2:

a) **The Lune property**: Let uv be any line of an SMT, and $L(u,v)$ be the region consisting of all points p satisfying, $|pu| < |uv|$ and $|pv| < |uv|$. Then, the lune-shaped area $L(u,v)$ contains no other vertex of SMT.

b) **The Wedge property**: Let W be any open wedge-shaped region having angle that is greater or equal to 120° and containing none of the terminals. Then W contains no Steiner points.

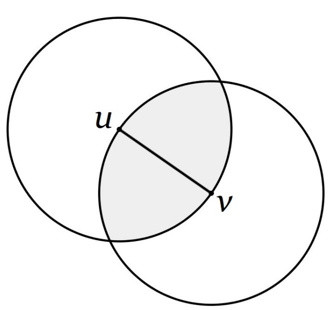


Figure 1: Lune property

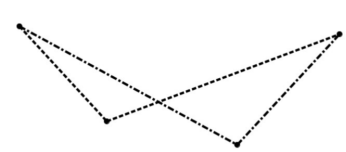


Figure 2: Wedge property

3.2 Constructing the Steiner Polygon

The Steiner polygon begins with connecting the convex hull of $\mathscr{H}$. For any two neighbor nodes n_i, n_j on $\mathscr{H}$, if there exists an adjacent node n_k, such that the area closed by three nodes contains no other nodes and each pair nodes have the internal angle $n_i n_k n_j$ greater or equal to 120°, then the edge $n_i n_j$ is replaced by edges $n_i n_k$ and $n_k n_j$ to give a new polygon $\mathscr{H}_{i+1}$. The process stops when no changes are available.

3.3 Steiner Topologies Degeneration

Cockayne[8] pointed out, for i nodes lie on a convex Steiner polygon, only $\frac{(n-1)!}{(i-1)!}$ cyclic permutations to consider.Besides, triangulation of $n-2$ sides by crossing-free diagonals is $(n-2)^{th}$ Catalan number that is $\binom{2n-4}{n-2}\frac{1}{n-1}$ which is approximately $\frac{4^{n-2}}{(n-2)^{3/2}\sqrt{\pi}}$ by Stirling's approximation formula. The total number of Steiner topologies is the product of those two numbers.

Note that, if $n-i$ is a constant, then we have reduced the rough upper-bound into an exponential number rather than super-exponential many. Actually, we still can explore more properties in Steiner polygon to further lower the upper-bound.

4 Special Steiner Tree

SMT in some special geometric configurations can be obtained though some thorough analysis. Most of the cases in this chapter can be solved in polynomial time, and we also conclude some cases with special and elegant properties.

4.1 Three Nodes

For the three points case, there is only one unique full Steiner topology existed. If FST does not exist, then SMT is equal to MST that is taking away the longest edge. Figure 5 and 6 gives two conditions for 3-nodes topologies.

The definition of Steiner point states that included angles between arcs connecting the two adjacent nodes are all 120 degree. Therefore, the two different Steiner topologies give the same Steiner topology structure. For the interior angle is greater than 120 degree, the segment does not intersection the arc that leaves the case degenerates into the MST. This case can also be simply proved by Wedge property.

Figure 3: Left: Full Steiner topology for 3-nodes, Right: Degeneration case for 3-nodes

4.2 Four Nodes

Steiner tree for three nodes is easily understood. However, a complete understanding for four nodes is a relatively recent thing. For considering all the generality, we assume that $[a, c]$ and $[b, d]$ are the two diagonals of the convex quadrilateral, and they intersect at point o. The two possible FST structures (as shown in Figure 7.) with vertex a and b adjacent to the same Steiner point, and the other with vertex a and d. Call them $ab-tree$ and $ad-tree$ respectively.

The first case we are going to consider is whether a FST exists. *Pollock et al.*[1] gave his result in the following lemma.

Lemma 4.1. *The FTP existed implies that the four nodes must form a convex quadrilateral.*

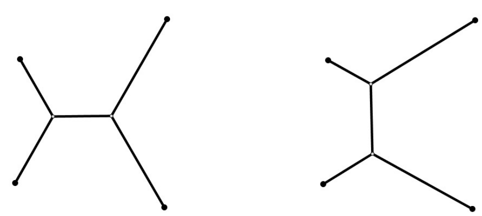


Figure 4: Two possible FSTs in a convex quadrilateral

Du et al.[9] gave necessary and sufficient conditions for the existence of $ad-tree$, similar conditions can be stated for the $ab-tree$.

Lemma 4.2. *Necessary and sufficient conditions for the existence of the $ad-tree$:*

1. *The quadrilateral $abcd$ is convex.*
2. $\angle da(bc)$, $\angle(bc)da$, $\angle(da)bc$, *and* $\angle bc(da)$ *are all less than* $120°$
3. $\angle aod < 120°$

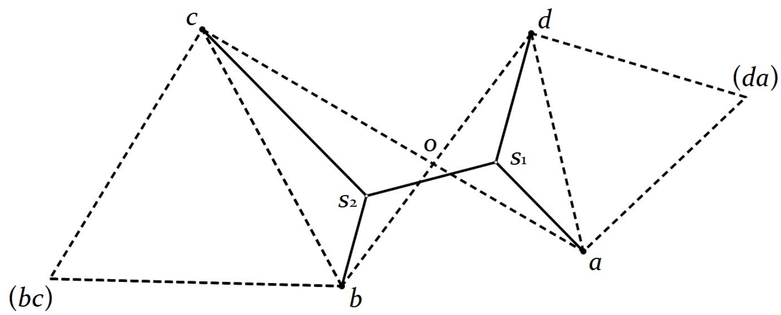


Figure 5: The $ad - tree$

Furthermore, we gave the sketch proof for the criterion of judging the optimal FST in two possible configurations.

Theorem 4.3. *Suppose that both FSTs exists. The one with a longer edge connecting the two Steiner points is the shorter tree.*

Proof. We can prove that Simson line of the acute FST is shorter than that of the obtuse tree regardless of whether the latter is exist or not. Then the proof shows that every non-full Steiner tree is at least as long as one of the Simson line.

□

Du et al. were motivated by the need to prove a decomposition theorem involving the removal of a quadrilateral from a convex Steiner hull. The following result obtained by them was crucially used improved the Wedge property as shown in Section 3.1.

Theorem 4.4. *Let $abcd$ be a convex quadrilateral with $\angle a$ and $\angle b \geq 120°$ and $\angle aob \geq \angle a + \angle b - 150°$. Let $a'b'c'd'$ be a quadrilateral embedded in $abcd$ with a', d' on $[a, d]$ and c', b' on $[c, b]$. Then an SMT for $\{a', b', c', d'\}$ cannot be full.*

4.3 Regular Polygon

Theorem 4.5. *When $n \geq 6$, the Steiner Minimum Tree for a regular n polygon is equivalent to Minimum Spanning Tree.*

Proof. Rubinstein and Thomas [11] proposed a variational approach used to prove results which do not depend on scaling, it is assumed that

$$\sum_{i=1}^{2n-3} y_i = 1.$$

The $(2n-4)$- dimensional simplex

$$\Delta(F) = \left\{ Y = \{y_1, y_2 \ldots, y_{2n-3}\} : \sum_{i=1}^{2n-3} y_i = 1, y_i \geq 0 \,\forall i \right\}$$

is called a configuration space, and element $Y \in \Delta(F)$ is called a configuration. Let V be a vector at Y. Define T to be an MST for Y such that $T(h)$ is also an MST for $Y + \Delta h V$ for Δh is sufficiently small. Let S be a SMT for Y with topology in $D_S(F)$. Then $\rho(V) = \frac{L_S(V)}{L_T(V)}$, where L_S and L_T are the lengths of the respective trees. Thus, we have,

$$\rho'(V) = \frac{L'_T(V)}{L_T(V)} \left[\frac{L'_S(V)}{L'_T(V)} - \rho \right]$$

We call V a reversible direction if reversing the direction of variation at ρ yields a change of signs of $Ł'_T(V)$ and $L'_S(V)$. This implies that $\rho'(V) = 0$, and consequently,

$$L'_S(V) = \rho L'_T(V)$$

if V is reversible. Furthermore, the second derivative of ρ also has a simple expression for reversible variations, i.e.,

$$\rho''(V) = \frac{L''_S(V) - \rho L''_T(V)}{L_T}$$

The strategy is to show that for every configuration at which ρ is minimal, a reversible variation can be constructed for which

$$L''_S(V) - \rho L''_T(V) < 0$$

Therefore $\rho''(V) < 0$, contradicting the minimality of ρ.

□

In fact, using the conditions on $\rho'(V)$ and $\rho''(V)$, we are able to show that if ρ achieves a minimum at the set $N = n_1, n_2, \ldots, n_m$ ordered around the circle, such that there exist an MST and an SMT, not identical. Then the following facts hold:

1. If $|n_i, n_{i+1}| > 1$, then $|n_{i-1}, n_i| = |n_{i+1}, n_{i+2}| = 1$.
2. Either $|n_{i-1}, n_i| \geq 1$ or $|n_i, n_{i+1}| \geq 1$ for $1 \leq i \leq m$.
3. There exists an $i, 0 \leq i \leq n-1$, such that $|n_i, n_{i+1}| \geq 1$.

From these fact, it can be immediately deduced that MST can have at move five edges of length ≥ 1, and at most five edges of length ≤ 1 A different approach for proving the Lemma 3.1 was proposed by *Hu, et al.*[7] using the results on Steiner ratio.

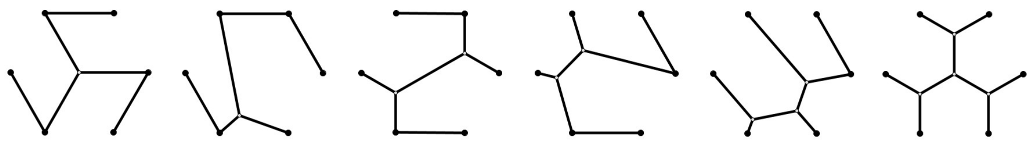


Figure 6: Different Steiner topologies in a regular hexagon

4.4 Grid Nodes

Let $[p_1, p_n]$ denote the segment connecting the two ends of a path H. Then $[p_1, p_n]$ and H together enclose a sequence of m polygons where two adjacent polygons meet at an intersection of $[p_1, p_n]$ and H. Let $T_i, i = 1, 2, \ldots, m$, denote a full SMT for the vertices of the i^{th} polygon. Then $\cup_{i=1}^{m} T_i$ is called a cut-and-patch tree for N. Most exciting results for co-path terminals are to determine conditions on the paths such that cut-and-patch trees are SMTs.

The first such results was given by *Chung and Graham*[12] for set of co-path terminals called ladders. Let L_n denote a ladder of $2n$ nodes arranged in a grid array forming a unit square. The path is an alternating sequence of vertical and horizontal segments wrapping around the squares in order.

Theorem 4.6. *A cut-and-patch tree for L_n is an SMT. Furthermore,*

$$|L_n| = \begin{cases} \sqrt{(n(1+\sqrt{3}/2)-1)^2 + 1/4} & \text{if } n \text{ is odd,} \\ n(1+\sqrt{3}/2)-1 & \text{if } n \text{ is even.} \end{cases}$$

For n is even, $[p_1, p_2n]$ is a side of the $2 \times n$ array and cuts H into $n/2$ 1×1 squares connected by $n/2 - 1$ unit segments. For n is odd, $[p_1, p_2n]$ is a diagonal of the $2 \times n$ rectangle and cuts H into $m - 1$ trapezoids. The cut-and-patch trees for $n = 4$ and 5 are shown in Figure 9.

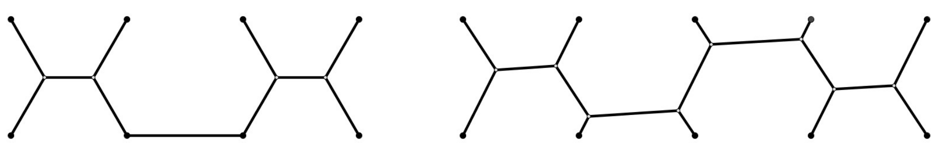
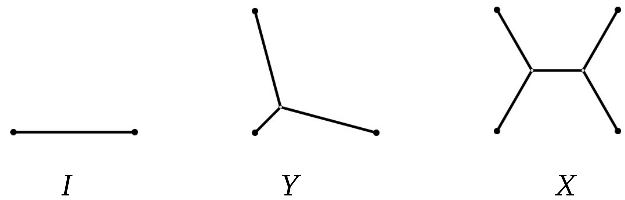


Figure 7: SMTs for ladder grids

Hwang et al.[13] studied SMTs when nodes are arranged in regular grids of unit squares. They gave proof about the constructions for 2×2 square. A basic building block denoted by X. In these constructions is a 4-point full SMT for the four corners of a unit square.

Theorem 4.7. *If a rectangular array can be spanned by an ST made up entirely of Xs, then the array is a square of size 2^t by 2^t for some $t \geq 1$.*



For square arrays of size $n = 2^t$, they gave a recursive construction of such an Steiner tree B_n by connecting four $B_{n/2}$ by an X in the center. Thus B_n consists of $(4^t - 1)/3X$s and $|B_n| = [(4^t - 1)/3](1 + \sqrt{3})$.

For square arrays of size $n \neq 2^t$, three other building blocks, I: a segment between two adjacent terminals, Y : 3-nodes full SMT on the tree corners of a unit square (shown in Figure 10.) and Z: a full SMT on a 2×5 ladder (shown in Figure 9.) are used to fill the gaps between Xs.

Let B_n denote the ST obtained by their recursive construction for an $n \times n$ array. We can conjecture the following table summarized the constructions for $n \neq 2^n$.

n	B_n	$\lvert B_n \rvert$
$6k$	$(12k^2 - 1)X + Y$	$(12k^2 - 1)(1 + \sqrt{3}) + (1 + \sqrt{3})/\sqrt{2}$
$6k + 1$	$(12k^2 + 4k - 1)X + 3I$	$(12k^2 + 4k - 1)(1 + \sqrt{3}) + 3$
$6k + 2$	$(12k^2 + 8k - 2)X + Z$	$(12k^2 + 8k - 2)(1 + \sqrt{3}) + \sqrt{35 + 20\sqrt{3}}$
$6k + 3$	$(12k^2 + 12k + 2)X + 2I$	$(12k^2 + 12k + 2)(1 + \sqrt{3}) + 2$
$6k + 4$	$(12k^2 + 16k + 2)X + Z$	$(12k^2 + 16k + 2)(1 + \sqrt{3}) + \sqrt{35 + 20\sqrt{3}}$
$6k + 5$	$(12k^2 + 20k + 7)X + 3I$	$(12k^2 + 20k + 7)(1 + \sqrt{3}) + 3$

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